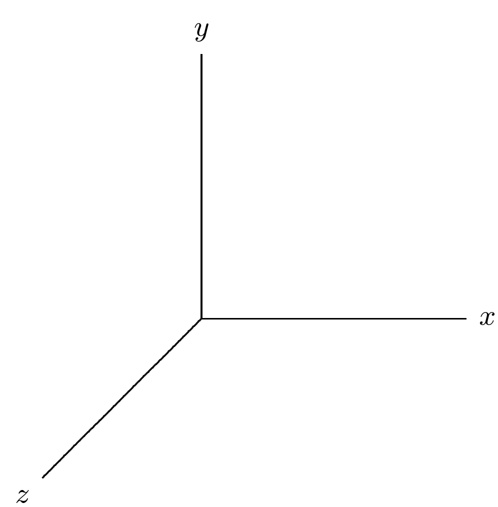
12

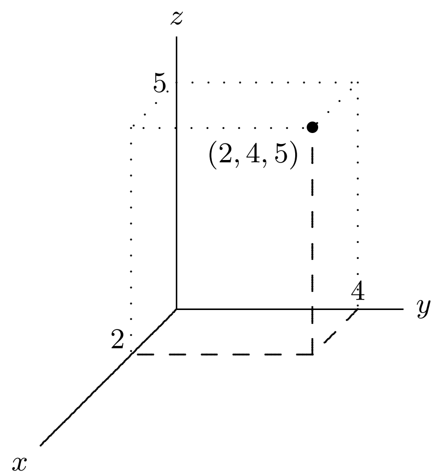
Three Dimensions

12.1 The Coordinate System

So far we have been investigating functions of the form $y = f(x)$, with one independent and one dependent variable. Such functions can be represented in two dimensions, using two numerical axes that allow us to identify every point in the plane with two numbers. We now want to talk about three-dimensional space; to identify every point in three dimensions we require three numerical values. The obvious way to make this association is to add one new axis, perpendicular to the x and y axes we already understand. We could, for example, add a third axis, the z axis, with the positive z axis coming straight out of the page, and the negative z axis going out the back of the page. This is difficult to work with on a printed page, so more often we draw a view of the three axes from an angle:



You must then imagine that the z axis is perpendicular to the other two. Just as we have investigated functions of the form $y = f(x)$ in two dimensions, we will investigate three dimensions largely by considering functions; now the functions will (typically) have the form $z = f(x, y)$. Because we are used to having the result of a function graphed in the vertical direction, it is somewhat easier to maintain that convention in three dimensions. To accomplish this, we normally rotate the axes so that z points up; the result is then:



Note that if you imagine looking down from above, along the z axis, the positive z axis will come straight toward you, the positive y axis will point up, and the positive x axis will point to your right, as usual. Any point in space is identified by providing the three coordinates of the point, as shown; naturally, we list the coordinates in the order (x, y, z). One useful way to think of this is to use the x and y coordinates to identify a point in the x-y plane, then move straight up (or down) a distance given by the z coordinate.

It is now fairly simple to understand some "shapes" in three dimensions that correspond to simple conditions on the coordinates. In two dimensions the equation $x = 1$ describes the vertical line through $(1, 0)$. In three dimensions, it still describes all points with x-coordinate 1, but this is now a plane, as in figure 12.1.1.

Recall the very useful distance formula in two dimensions: the distance between points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$; this comes directly from the Pythagorean theorem. What is the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in three dimensions? Geometrically, we want the length of the long diagonal labeled c in the "box" in figure 12.1.2. Since a, b, c form a right triangle, $a^2 + b^2 = c^2$. b is the vertical distance between (x_1, y_1, z_1) and (x_2, y_2, z_2), so $b = |z_1 - z_2|$. The length a runs parallel to the x-y plane, so it is simply the distance between (x_1, y_1) and (x_2, y_2), that is, $a^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$. Now we see that $c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ and $c = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

It is sometimes useful to give names to points, for example we might let $P_1 = (x_1, y_1, z_1)$, or more concisely we might refer to the point $P_1(x_1, y_1, z_1)$, and subsequently

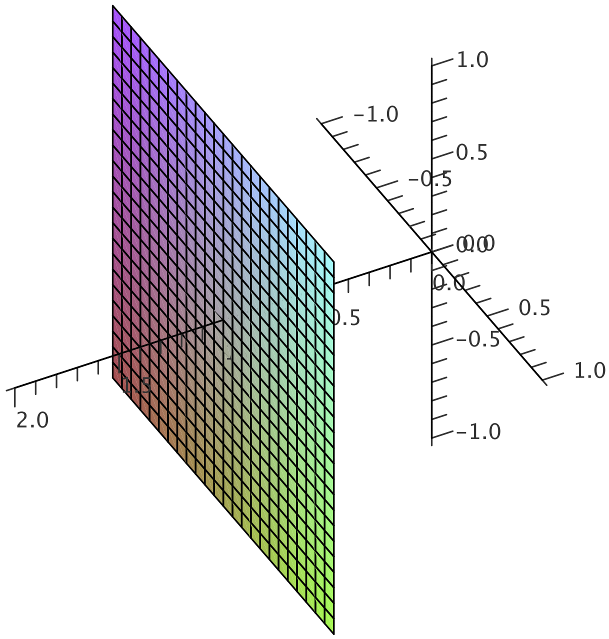


Figure 12.1.1 The plane $x = 1$. (AP)

use just P_1. Distance between two points in either two or three dimensions is sometimes denoted by d, so for example the formula for the distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ might be expressed as

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

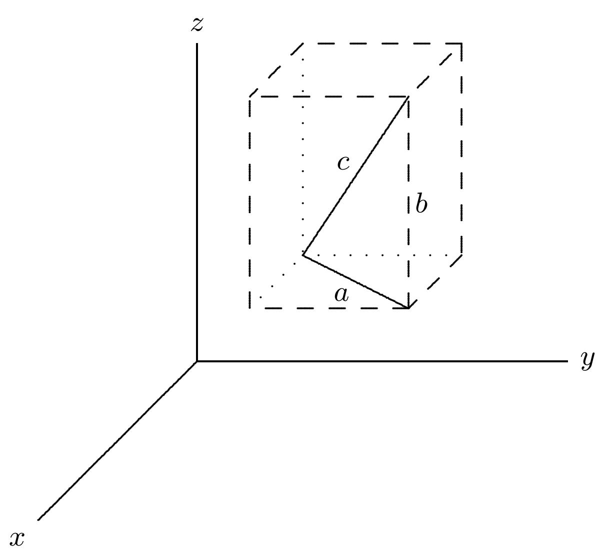


Figure 12.1.2 Distance in three dimensions.

In two dimensions, the distance formula immediately gives us the equation of a circle: the circle of radius r and center at (h, k) consists of all points (x, y) at distance r from

(h,k), so the equation is $r = \sqrt{(x-h)^2 + (y-k)^2}$ or $r^2 = (x-h)^2 + (y-k)^2$. Now we can get the similar equation $r^2 = (x-h)^2 + (y-k)^2 + (z-l)^2$, which describes all points (x,y,z) at distance r from (h,k,l), namely, the sphere with radius r and center (h,k,l).

Exercises 12.1.

1. Sketch the location of the points $(1,1,0)$, $(2,3,-1)$, and $(-1,2,3)$ on a single set of axes.

2. Describe geometrically the set of points (x,y,z) that satisfy $z = 4$.

3. Describe geometrically the set of points (x,y,z) that satisfy $y = -3$.

4. Describe geometrically the set of points (x,y,z) that satisfy $x + y = 2$.

5. The equation $x + y + z = 1$ describes some collection of points in $\mathbb{R}^3$. Describe and sketch the points that satisfy $x + y + z = 1$ and are in the x-y plane, in the x-z plane, and in the y-z plane.

6. Find the lengths of the sides of the triangle with vertices $(1,0,1)$, $(2,2,-1)$, and $(-3,2,-2)$. $\Rightarrow$

7. Find the lengths of the sides of the triangle with vertices $(2,2,3)$, $(8,6,5)$, and $(-1,0,2)$. Why do the results tell you that this isn't really a triangle? $\Rightarrow$

8. Find an equation of the sphere with center at $(1,1,1)$ and radius 2. $\Rightarrow$

9. Find an equation of the sphere with center at $(2,-1,3)$ and radius 5. $\Rightarrow$

10. Find an equation of the sphere with center $(3,-2,1)$ and that goes through the point $(4,2,5)$.

11. Find an equation of the sphere with center at $(2,1,-1)$ and radius 4. Find an equation for the intersection of this sphere with the y-z plane; describe this intersection geometrically. $\Rightarrow$

12. Consider the sphere of radius 5 centered at $(2,3,4)$. What is the intersection of this sphere with each of the coordinate planes?

13. Show that for all values of θ and ϕ, the point $(a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)$ lies on the sphere given by $x^2 + y^2 + z^2 = a^2$.

14. Prove that the midpoint of the line segment connecting (x_1,y_1,z_1) to (x_2,y_2,z_2) is at $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}, \dfrac{z_1+z_2}{2}\right)$.

15. Any three points $P_1(x_1,y_1,z_1)$, $P_2(x_2,y_2,z_2)$, $P_3(x_3,y_3,z_3)$, lie in a plane and form a triangle. The **triangle inequality** says that $d(P_1,P_3) \le d(P_1,P_2) + d(P_2,P_3)$. Prove the triangle inequality using either algebra (messy) or the law of cosines (less messy).

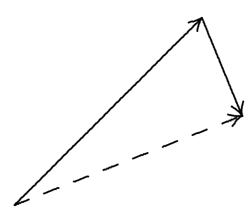
16. Is it possible for a plane to intersect a sphere in exactly two points? Exactly one point? Explain.

12.2 Vectors

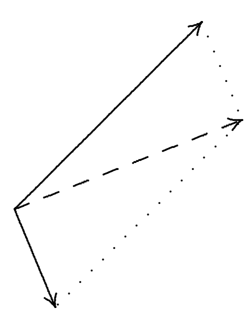
A vector is a quantity consisting of a non-negative magnitude and a direction. We could represent a vector in two dimensions as (m,θ), where m is the magnitude and θ is the direction, measured as an angle from some agreed upon direction. For example, we might think of the vector $(5, 45^\circ)$ as representing "5 km toward the northeast"; that is, this vector might be a **displacement vector**, indicating, say, that your grandfather walked

5 kilometers toward the northeast to school in the snow. On the other hand, the same vector could represent a velocity, indicating that your grandfather walked at 5 km/hr toward the northeast. What the vector does not indicate is where this walk occurred: a vector represents a magnitude and a direction, but not a location. Pictorially it is useful to represent a vector as an arrow; the direction of the vector, naturally, is the direction in which the arrow points; the magnitude of the vector is reflected in the length of the arrow.

It turns out that many, many quantities behave as vectors, e.g., displacement, velocity, acceleration, force. Already we can get some idea of their usefulness using displacement vectors. Suppose that your grandfather walked 5 km NE and then 2 km SSE; if the terrain allows, and perhaps armed with a compass, how could your grandfather have walked directly to his destination? We can use vectors (and a bit of geometry) to answer this question. We begin by noting that since vectors do not include a specification of position, we can "place" them anywhere that is convenient. So we can picture your grandfather's journey as two displacement vectors drawn head to tail:



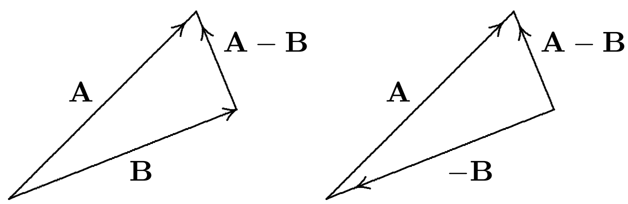
The displacement vector for the shortcut route is the vector drawn with a dashed line, from the tail of the first to the head of the second. With a little trigonometry, we can compute that the third vector has magnitude approximately 4.62 and direction 21.43°, so walking 4.62 km in the direction 21.43° north of east (approximately ENE) would get your grandfather to school. This sort of calculation is so common, we dignify it with a name: we say that the third vector is the **sum** of the other two vectors. There is another common way to picture the sum of two vectors. Put the vectors tail to tail and then complete the parallelogram they indicate; the sum of the two vectors is the diagonal of the parallelogram:



This is a more natural representation in some circumstances. For example, if the two original vectors represent forces acting on an object, the sum of the two vectors is the net or effective force on the object, and it is nice to draw all three with their tails at the location of the object.

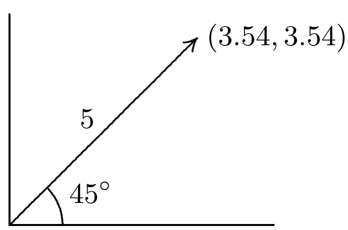
We also define **scalar multiplication** for vectors: if $\mathbf{A}$ is a vector (m, θ) and $a \geq 0$ is a real number, the vector $a\mathbf{A}$ is (am, θ), namely, it points in the same direction but has a times the magnitude. If $a < 0$, $a\mathbf{A}$ is $(|a|m, \theta + \pi)$, with $|a|$ times the magnitude and pointing in the opposite direction (unless we specify otherwise, angles are measured in radians).

Now we can understand subtraction of vectors: $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B}$:

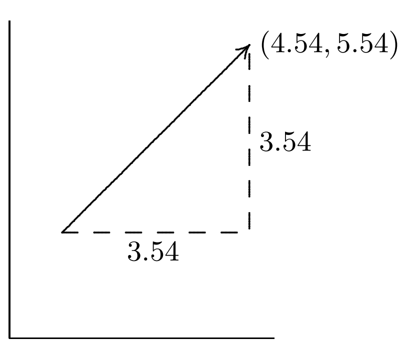


Note that as you would expect, $\mathbf{B} + (\mathbf{A} - \mathbf{B}) = \mathbf{A}$.

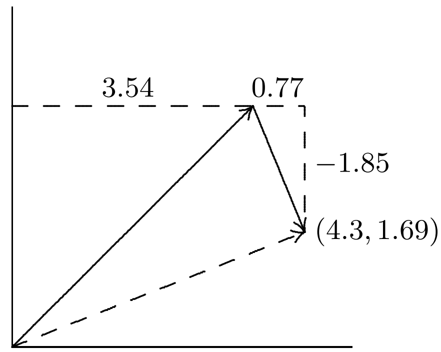
We can represent a vector in ways other than (m, θ), and in fact (m, θ) is not generally used at all. How else could we describe a particular vector? Consider again the vector $(5, 45^\circ)$. Let's draw it again, but impose a coordinate system. If we put the tail of the arrow at the origin, the head of the arrow ends up at the point $(5/\sqrt{2}, 5/\sqrt{2}) \approx (3.54, 3.54)$.



In this picture the coordinates $(3.54, 3.54)$ identify the head of the arrow, provided we know that the tail of the arrow has been placed at $(0, 0)$. Then in fact the vector can always be identified as $(3.54, 3.54)$, no matter where it is placed; we just have to remember that the numbers 3.54 must be interpreted as a *change* from the position of the tail, not as the actual coordinates of the arrow head; to emphasize this we will write $\langle 3.54, 3.54\rangle$ to mean the vector and $(3.54, 3.54)$ to mean the point. Then if the vector $\langle 3.54, 3.54\rangle$ is drawn with its tail at $(1, 2)$ it looks like this:



Consider again the two part trip: 5 km NE and then 2 km SSE. The vector representing the first part of the trip is $\langle 5/\sqrt{2}, 5/\sqrt{2}\rangle$, and the second part of the trip is represented by $\langle 2\cos(-3\pi/8), 2\sin(-3\pi/8)\rangle \approx \langle 0.77, -1.85\rangle$. We can represent the sum of these with the usual head to tail picture:



It is clear from the picture that the coordinates of the destination point are $(5/\sqrt{2} + 2\cos(-3\pi/8), 5/\sqrt{2} + 2\sin(-3\pi/8))$ or approximately $(4.3, 1.69)$, so the sum of the two vectors is $\langle 5/\sqrt{2} + 2\cos(-3\pi/8), 5/\sqrt{2} + 2\sin(-3\pi/8)\rangle \approx \langle 4.3, 1.69\rangle$. Adding the two vectors is easier in this form than in the (m, θ) form, provided that we're willing to have the answer in this form as well.

It is easy to see that scalar multiplication and vector subtraction are also easy to compute in this form: $a\langle v, w\rangle = \langle av, aw\rangle$ and $\langle v_1, w_1\rangle - \langle v_2, w_2\rangle = \langle v_1 - v_2, w_1 - w_2\rangle$. What about the magnitude? The magnitude of the vector $\langle v, w\rangle$ is still the length of the corresponding arrow representation; this is the distance from the origin to the point (v, w), namely, the distance from the tail to the head of the arrow. We know how to compute distances, so the magnitude of the vector is simply $\sqrt{v^2 + w^2}$, which we also denote with absolute value bars: $|\langle v, w\rangle| = \sqrt{v^2 + w^2}$.

In three dimensions, vectors are still quantities consisting of a magnitude and a direction, but of course there are many more possible directions. It's not clear how we might represent the direction explicitly, but the coordinate version of vectors makes just as much sense in three dimensions as in two. By $\langle 1, 2, 3\rangle$ we mean the vector whose head is at $(1, 2, 3)$ if its tail is at the origin. As before, we can place the vector anywhere we want; if it has its tail at $(4, 5, 6)$ then its head is at $(5, 7, 9)$. It remains true that arithmetic is easy to do with vectors in this form:

$$a\langle v_1, v_2, v_3\rangle = \langle av_1, av_2, av_3\rangle$$
$$\langle v_1, v_2, v_3\rangle + \langle w_1, w_2, w_3\rangle = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3\rangle$$
$$\langle v_1, v_2, v_3\rangle - \langle w_1, w_2, w_3\rangle = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3\rangle$$

The magnitude of the vector is again the distance from the origin to the head of the arrow, or $|\langle v_1, v_2, v_3\rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

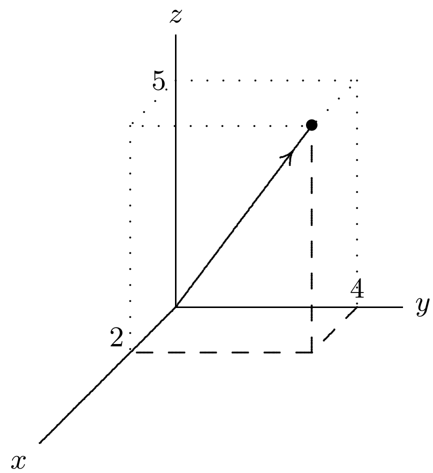


Figure 12.2.1 The vector $\langle 2,4,5\rangle$ with its tail at the origin.

Three particularly simple vectors turn out to be quite useful: $\mathbf{i}=\langle 1,0,0\rangle$, $\mathbf{j}=\langle 0,1,0\rangle$, and $\mathbf{k}=\langle 0,0,1\rangle$. These play much the same role for vectors that the axes play for points. In particular, notice that

$$\begin{aligned}\langle v_1,v_2,v_3\rangle &= \langle v_1,0,0\rangle+\langle 0,v_2,0\rangle+\langle 0,0,v_3\rangle\\ &= v_1\langle 1,0,0\rangle+v_2\langle 0,1,0\rangle+v_3\langle 0,0,1\rangle\\ &= v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}\end{aligned}$$

We will frequently want to produce a vector that points from one point to another. That is, if P and Q are points, we seek the vector $\mathbf{x}$ such that when the tail of $\mathbf{x}$ is placed at P, its head is at Q; we refer to this vector as $\overrightarrow{PQ}$. If we know the coordinates of P and Q, the coordinates of the vector are easy to find.

EXAMPLE 12.2.1 Suppose $P=(1,-2,4)$ and $Q=(-2,1,3)$. The vector $\overrightarrow{PQ}$ is $\langle -2-1,1--2,3-4\rangle=\langle -3,3,-1\rangle$ and $\overrightarrow{QP}=\langle 3,-3,1\rangle$. □

Exercises 12.2.

1. Draw the vector $\langle 3,-1\rangle$ with its tail at the origin.
2. Draw the vector $\langle 3,-1,2\rangle$ with its tail at the origin.
3. Let $\mathbf{A}$ be the vector with tail at the origin and head at $(1,2)$; let $\mathbf{B}$ be the vector with tail at the origin and head at $(3,1)$. Draw $\mathbf{A}$ and $\mathbf{B}$ and a vector $\mathbf{C}$ with tail at $(1,2)$ and head at $(3,1)$. Draw $\mathbf{C}$ with its tail at the origin.
4. Let $\mathbf{A}$ be the vector with tail at the origin and head at $(-1,2)$; let $\mathbf{B}$ be the vector with tail at the origin and head at $(3,3)$. Draw $\mathbf{A}$ and $\mathbf{B}$ and a vector $\mathbf{C}$ with tail at $(-1,2)$ and head at $(3,3)$. Draw $\mathbf{C}$ with its tail at the origin.

5. Let $\mathbf{A}$ be the vector with tail at the origin and head at $(5, 2)$; let $\mathbf{B}$ be the vector with tail at the origin and head at $(1, 5)$. Draw $\mathbf{A}$ and $\mathbf{B}$ and a vector $\mathbf{C}$ with tail at $(5, 2)$ and head at $(1, 5)$. Draw $\mathbf{C}$ with its tail at the origin.

6. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v} = \langle 1, 3\rangle$ and $\mathbf{w} = \langle -1, -5\rangle$. ⇒

7. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v} = \langle 1, 2, 3\rangle$ and $\mathbf{w} = \langle -1, 2, -3\rangle$. ⇒

8. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v} = \langle 1, 0, 1\rangle$ and $\mathbf{w} = \langle -1, -2, 2\rangle$. ⇒

9. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v} = \langle 1, -1, 1\rangle$ and $\mathbf{w} = \langle 0, 0, 3\rangle$. ⇒

10. Find $|\mathbf{v}|$, $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $|\mathbf{v}+\mathbf{w}|$, $|\mathbf{v}-\mathbf{w}|$ and $-2\mathbf{v}$ for $\mathbf{v} = \langle 3, 2, 1\rangle$ and $\mathbf{w} = \langle -1, -1, -1\rangle$. ⇒

11. Let $P = (4, 5, 6)$, $Q = (1, 2, -5)$. Find $\overrightarrow{PQ}$. Find a vector with the same direction as $\overrightarrow{PQ}$ but with length 1. Find a vector with the same direction as $\overrightarrow{PQ}$ but with length 4. ⇒

12. If A, B, and C are three points, find $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$. ⇒

13. Consider the 12 vectors that have their tails at the center of a clock and their respective heads at each of the 12 digits. What is the sum of these vectors? What if we remove the vector corresponding to 4 o'clock? What if, instead, all vectors have their tails at 12 o'clock, and their heads on the remaining digits? ⇒

14. Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero vectors in two dimensions that are not parallel or anti-parallel. Show, algebraically, that if $\mathbf{c}$ is any two dimensional vector, there are scalars s and t such that $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.

15. Does the statement in the previous exercise hold if the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are three dimensional vectors? Explain.

12.3 The Dot Product

Here's a question whose answer turns out to be very useful: Given two vectors, what is the angle between them?

It may not be immediately clear that the question makes sense, but it's not hard to turn it into a question that does. Since vectors have no position, we are as usual free to place vectors wherever we like. If the two vectors are placed tail-to-tail, there is now a reasonable interpretation of the question: we seek the measure of the smallest angle between the two vectors, in the plane in which they lie. Figure 12.3.1 illustrates the situation.

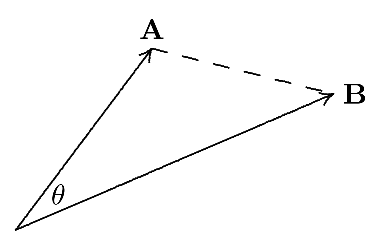


Figure 12.3.1 The angle between vectors $\mathbf{A}$ and $\mathbf{B}$.

Since the angle θ lies in a triangle, we can compute it using a bit of trigonometry, namely, the law of cosines. The lengths of the sides of the triangle in figure 12.3.1 are $|\mathbf{A}|$,

$|\mathbf{B}|$, and $|\mathbf{A}-\mathbf{B}|$. Let $\mathbf{A}=\langle a_1,a_2,a_3\rangle$ and $\mathbf{B}=\langle b_1,b_2,b_3\rangle$; then

$$\begin{aligned}
|\mathbf{A}-\mathbf{B}|^2 &= |\mathbf{A}|^2+|\mathbf{B}|^2-2|\mathbf{A}||\mathbf{B}|\cos\theta\\
2|\mathbf{A}||\mathbf{B}|\cos\theta &= |\mathbf{A}|^2+|\mathbf{B}|^2-|\mathbf{A}-\mathbf{B}|^2\\
&= a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2-(a_1-b_1)^2-(a_2-b_2)^2-(a_3-b_3)^2\\
&= a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2\\
&\qquad -(a_1^2-2a_1b_1+b_1^2)-(a_2^2-2a_2b_2+b_2^2)-(a_3^2-2a_3b_3+b_3^2)\\
&= 2a_1b_1+2a_2b_2+2a_3b_3\\
|\mathbf{A}||\mathbf{B}|\cos\theta &= a_1b_1+a_2b_2+a_3b_3\\
\cos\theta &= (a_1b_1+a_2b_2+a_3b_3)/(|\mathbf{A}||\mathbf{B}|)
\end{aligned}$$

So a bit of simple arithmetic with the coordinates of $\mathbf{A}$ and $\mathbf{B}$ allows us to compute the cosine of the angle between them. If necessary we can use the arccosine to get θ, but in many problems $\cos\theta$ turns out to be all we really need.

The numerator of the fraction that gives us $\cos\theta$ turns up a lot, so we give it a name and more compact notation: we call it the **dot product**, and write it as

$$\mathbf{A}\cdot\mathbf{B}=a_1b_1+a_2b_2+a_3b_3.$$

This is the same symbol we use for ordinary multiplication, but there should never be any confusion; you can tell from context whether we are "multiplying" vectors or numbers. (We might also use the dot for scalar multiplication: $a\cdot\mathbf{V}=a\mathbf{V}$; again, it is clear what is meant from context.)

EXAMPLE 12.3.1 Find the angle between the vectors $\mathbf{A}=\langle 1,2,1\rangle$ and $\mathbf{B}=\langle 3,1,-5\rangle$. We know that $\cos\theta=\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|)=(1\cdot 3+2\cdot 1+1\cdot(-5))/(|\mathbf{A}||\mathbf{B}|)=0$, so $\theta=\pi/2$, that is, the vectors are perpendicular. □

EXAMPLE 12.3.2 Find the angle between the vectors $\mathbf{A}=\langle 3,3,0\rangle$ and $\mathbf{B}=\langle 1,0,0\rangle$. We compute

$$\begin{aligned}
\cos\theta &= (3\cdot 1+3\cdot 0+0\cdot 0)/(\sqrt{9+9+0}\sqrt{1+0+0})\\
&= 3/\sqrt{18}=1/\sqrt{2}
\end{aligned}$$

so $\theta=\pi/4$. □

EXAMPLE 12.3.3 Some special cases are worth looking at: Find the angles between $\mathbf{A}$ and $\mathbf{A}$; $\mathbf{A}$ and $-\mathbf{A}$; $\mathbf{A}$ and $\mathbf{0}=\langle 0,0,0\rangle$.

$\cos\theta = \mathbf{A}\cdot\mathbf{A}/(|\mathbf{A}||\mathbf{A}|) = (a_1^2+a_2^2+a_3^2)/(\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{a_1^2+a_2^2+a_3^2}) = 1$, so the angle between $\mathbf{A}$ and itself is zero, which of course is correct.

$\cos\theta = \mathbf{A}\cdot -\mathbf{A}/(|\mathbf{A}||-\mathbf{A}|) = (-a_1^2-a_2^2-a_3^2)/(\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{a_1^2+a_2^2+a_3^2}) = -1$, so the angle is π, that is, the vectors point in opposite directions, as of course we already knew.

$\cos\theta = \mathbf{A}\cdot\mathbf{0}/(|\mathbf{A}||\mathbf{0}|) = (0+0+0)/(\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{0^2+0^2+0^2})$, which is undefined. On the other hand, note that since $\mathbf{A}\cdot\mathbf{0}=0$ it looks at first as if $\cos\theta$ will be zero, which as we have seen means that vectors are perpendicular; only when we notice that the denominator is also zero do we run into trouble. One way to "fix" this is to adopt the convention that the zero vector $\mathbf{0}$ is perpendicular to all vectors; then we can say in general that if $\mathbf{A}\cdot\mathbf{B}=0$, $\mathbf{A}$ and $\mathbf{B}$ are perpendicular. □

Generalizing the examples, note the following useful facts:

1. If $\mathbf{A}$ is parallel or anti-parallel to $\mathbf{B}$ then $\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|)=\pm1$, and conversely, if $\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|)=1$, $\mathbf{A}$ and $\mathbf{B}$ are parallel, while if $\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|)=-1$, $\mathbf{A}$ and $\mathbf{B}$ are anti-parallel. (Vectors are parallel if they point in the same direction, anti-parallel if they point in opposite directions.)

2. If $\mathbf{A}$ is perpendicular to $\mathbf{B}$ then $\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|)=0$, and conversely if $\mathbf{A}\cdot\mathbf{B}/(|\mathbf{A}||\mathbf{B}|)=0$ then $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

Given two vectors, it is often useful to find the **projection** of one vector onto the other, because this turns out to have important meaning in many circumstances. More precisely, given $\mathbf{A}$ and $\mathbf{B}$, we seek a vector parallel to $\mathbf{B}$ but with length determined by $\mathbf{A}$ in a natural way, as shown in figure 12.3.2. $\mathbf{V}$ is chosen so that the triangle formed by $\mathbf{A}$, $\mathbf{V}$, and $\mathbf{A}-\mathbf{V}$ is a right triangle.

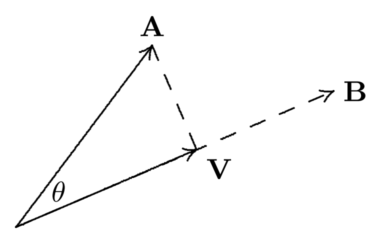


Figure 12.3.2 $\mathbf{V}$ is the projection of $\mathbf{A}$ onto $\mathbf{B}$.

Using a little trigonometry, we see that

$$|\mathbf{V}| = |\mathbf{A}|\cos\theta = |\mathbf{A}|\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|} = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|};$$

this is sometimes called the **scalar projection of A onto B**. To get $\mathbf{V}$ itself, we multiply this length by a vector of length one parallel to $\mathbf{B}$:

$$\mathbf{V} = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|}\frac{\mathbf{B}}{|\mathbf{B}|} = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}.$$

Be sure that you understand why $\mathbf{B}/|\mathbf{B}|$ is a vector of length one (also called a **unit vector**) parallel to $\mathbf{B}$.

The discussion so far implicitly assumed that $0 \le \theta \le \pi/2$. If $\pi/2 < \theta \le \pi$, the picture is like figure 12.3.3. In this case $\mathbf{A}\cdot\mathbf{B}$ is negative, so the vector

$$\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|^2}\mathbf{B}$$

is anti-parallel to $\mathbf{B}$, and its length is

$$\left|\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{B}|}\right|.$$

So in general, the scalar projection of $\mathbf{A}$ onto $\mathbf{B}$ may be positive or negative. If it is negative, it means that the projection vector is anti-parallel to $\mathbf{B}$ and that the length of the projection vector is the absolute value of the scalar projection. Of course, you can also compute the length of the projection vector as usual, by applying the distance formula to the vector.

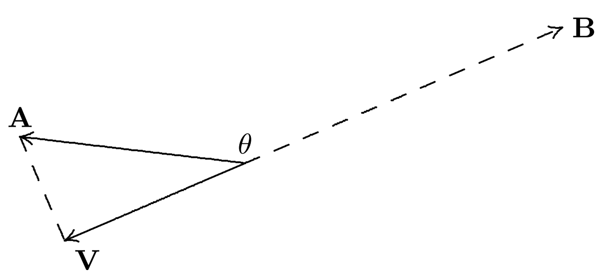


Figure 12.3.3 $\mathbf{V}$ is the projection of $\mathbf{A}$ onto $\mathbf{B}$.

Note that the phrase "projection onto $\mathbf{B}$" is a bit misleading if taken literally; all that $\mathbf{B}$ provides is a direction; the length of $\mathbf{B}$ has no impact on the final vector. In figure 12.3.4, for example, $\mathbf{B}$ is shorter than the projection vector, but this is perfectly acceptable.

Figure 12.3.4 $\mathbf{V}$ is the projection of $\mathbf{A}$ onto $\mathbf{B}$.

EXAMPLE 12.3.4 Physical force is a vector quantity. It is often necessary to compute the "component" of a force acting in a different direction than the force is being applied. For example, suppose a ten pound weight is resting on an inclined plane—a pitched roof, for example. Gravity exerts a force of ten pounds on the object, directed straight down. It is useful to think of the component of this force directed down and parallel to the roof, and the component down and directly into the roof. These forces are the projections of the force vector onto vectors parallel and perpendicular to the roof. Suppose the roof is tilted at a 30° angle, as in figure 12.3.5. A vector parallel to the roof is $\langle -\sqrt{3}, -1\rangle$, and a vector perpendicular to the roof is $\langle 1, -\sqrt{3}\rangle$. The force vector is $\mathbf{F} = \langle 0, -10\rangle$. The component of the force directed down the roof is then

$$\mathbf{F}_1 = \frac{\mathbf{F}\cdot\langle -\sqrt{3}, -1\rangle}{|\langle -\sqrt{3}, -1\rangle|^2}\langle -\sqrt{3}, -1\rangle = \frac{10}{2}\frac{\langle -\sqrt{3}, -1\rangle}{2} = \langle -5\sqrt{3}/2, -5/2\rangle$$

with length 5. The component of the force directed into the roof is

$$\mathbf{F}_2 = \frac{\mathbf{F}\cdot\langle 1, -\sqrt{3}\rangle}{|\langle 1, -\sqrt{3}\rangle|^2}\langle 1, -\sqrt{3}\rangle = \frac{10\sqrt{3}}{2}\frac{\langle 1, -\sqrt{3}\rangle}{2} = \langle 5\sqrt{3}/2, -15/2\rangle$$

with length $5\sqrt{3}$. Thus, a force of 5 pounds is pulling the object down the roof, while a force of $5\sqrt{3}$ pounds is pulling the object into the roof. □

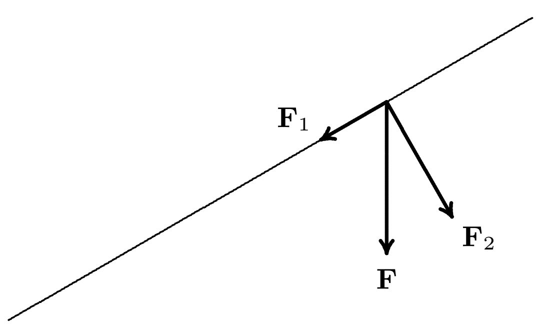


Figure 12.3.5 Components of a force.

The dot product has some familiar-looking properties that will be useful later, so we list them here. These may be proved by writing the vectors in coordinate form and then performing the indicated calculations; subsequently it can be easier to use the properties instead of calculating with coordinates.

THEOREM 12.3.5 If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and a is a real number, then

1. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$

2. $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$

3. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

4. $(a\mathbf{u})\cdot\mathbf{v}=a(\mathbf{u}\cdot\mathbf{v})=\mathbf{u}\cdot(a\mathbf{v})$ ■

Exercises 12.3.

1. Find $\langle 1,1,1\rangle\cdot\langle 2,-3,4\rangle$. ⇒

2. Find $\langle 1,2,0\rangle\cdot\langle 0,0,57\rangle$. ⇒

3. Find $\langle 3,2,1\rangle\cdot\langle 0,1,0\rangle$. ⇒

4. Find $\langle -1,-2,5\rangle\cdot\langle 1,0,-1\rangle$. ⇒

5. Find $\langle 3,4,6\rangle\cdot\langle 2,3,4\rangle$. ⇒

6. Find the cosine of the angle between $\langle 1,2,3\rangle$ and $\langle 1,1,1\rangle$; use a calculator if necessary to find the angle. ⇒

7. Find the cosine of the angle between $\langle -1,-2,-3\rangle$ and $\langle 5,0,2\rangle$; use a calculator if necessary to find the angle. ⇒

8. Find the cosine of the angle between $\langle 47,100,0\rangle$ and $\langle 0,0,5\rangle$; use a calculator if necessary to find the angle. ⇒

9. Find the cosine of the angle between $\langle 1,0,1\rangle$ and $\langle 0,1,1\rangle$; use a calculator if necessary to find the angle. ⇒

10. Find the cosine of the angle between $\langle 2,0,0\rangle$ and $\langle -1,1,-1\rangle$; use a calculator if necessary to find the angle. ⇒

11. Find the angle between the diagonal of a cube and one of the edges adjacent to the diagonal. ⇒

12. Find the scalar and vector projections of $\langle 1,2,3\rangle$ onto $\langle 1,2,0\rangle$. ⇒

13. Find the scalar and vector projections of $\langle 1,1,1\rangle$ onto $\langle 3,2,1\rangle$. ⇒

14. A force of 10 pounds is applied to a wagon, directed at an angle of 30°. Find the component of this force pulling the wagon straight up, and the component pulling it horizontally along the ground. ⇒

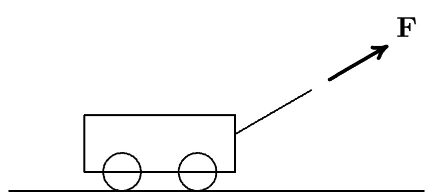


Figure 12.3.6 Pulling a wagon.

15. A force of 15 pounds is applied to a wagon, directed at an angle of 45°. Find the component of this force pulling the wagon straight up, and the component pulling it horizontally along the ground. ⇒

16. Use the dot product to find a non-zero vector $\mathbf{w}$ perpendicular to both $\mathbf{u}=\langle 1,2,-3\rangle$ and $\mathbf{v}=\langle 2,0,1\rangle$. ⇒

17. Let $\mathbf{x} = \langle 1,1,0\rangle$ and $\mathbf{y} = \langle 2,4,2\rangle$. Find a unit vector that is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$. $\Rightarrow$

18. Do the three points $(1,2,0)$, $(-2,1,1)$, and $(0,3,-1)$ form a right triangle? $\Rightarrow$

19. Do the three points $(1,1,1)$, $(2,3,2)$, and $(5,0,-1)$ form a right triangle? $\Rightarrow$

20. Show that $|\mathbf{A}\cdot\mathbf{B}| \le |\mathbf{A}||\mathbf{B}|$

21. Let $\mathbf{x}$ and $\mathbf{y}$ be perpendicular vectors. Use Theorem 12.3.5 to prove that $|\mathbf{x}|^2+|\mathbf{y}|^2 = |\mathbf{x}+\mathbf{y}|^2$. What is this result better known as?

22. Prove that the diagonals of a rhombus intersect at right angles.

23. Suppose that $\mathbf{z} = |\mathbf{x}|\mathbf{y} + |\mathbf{y}|\mathbf{x}$ where $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are all nonzero vectors. Prove that $\mathbf{z}$ bisects the angle between $\mathbf{x}$ and $\mathbf{y}$.

24. Prove Theorem 12.3.5.

12.4 The Cross Product

Another useful operation: Given two vectors, find a third vector perpendicular to the first two. There are of course an infinite number of such vectors of different lengths. Nevertheless, let us find one. Suppose $\mathbf{A} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3\rangle$. We want to find a vector $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ with $\mathbf{v}\cdot\mathbf{A} = \mathbf{v}\cdot\mathbf{B} = 0$, or

$$\begin{aligned} a_1v_1 + a_2v_2 + a_3v_3 &= 0,\\ b_1v_1 + b_2v_2 + b_3v_3 &= 0. \end{aligned}$$

Multiply the first equation by b_3 and the second by a_3 and subtract to get

$$\begin{aligned} b_3a_1v_1 + b_3a_2v_2 + b_3a_3v_3 &= 0\\ a_3b_1v_1 + a_3b_2v_2 + a_3b_3v_3 &= 0\\ (a_1b_3 - b_1a_3)v_1 + (a_2b_3 - b_2a_3)v_2 &= 0 \end{aligned}$$

Of course, this equation in two variables has many solutions; a particularly easy one to see is $v_1 = a_2b_3 - b_2a_3$, $v_2 = b_1a_3 - a_1b_3$. Substituting back into either of the original equations and solving for v_3 gives $v_3 = a_1b_2 - b_1a_2$.

This particular answer to the problem turns out to have some nice properties, and it is dignified with a name: the **cross product**:

$$\mathbf{A}\times\mathbf{B} = \langle a_2b_3 - b_2a_3, b_1a_3 - a_1b_3, a_1b_2 - b_1a_2\rangle.$$

While there is a nice pattern to this vector, it can be a bit difficult to memorize; here is a convenient mnemonic. The determinant of a two by two matrix is

$$\begin{vmatrix} a & b\\ c & d \end{vmatrix} = ad - cb.$$

This is extended to the determinant of a three by three matrix:

$$\begin{aligned}\begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} &= x\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - y\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + z\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ &= x(a_2b_3 - b_2a_3) - y(a_1b_3 - b_1a_3) + z(a_1b_2 - b_1a_2) \\ &= x(a_2b_3 - b_2a_3) + y(b_1a_3 - a_1b_3) + z(a_1b_2 - b_1a_2).\end{aligned}$$

Each of the two by two matrices is formed by deleting the top row and one column of the three by three matrix; the subtraction of the middle term must also be memorized. This is not the place to extol the uses of the determinant; suffice it to say that determinants are extraordinarily useful and important. Here we want to use it merely as a mnemonic device. You will have noticed that the three expressions in parentheses on the last line are precisely the three coordinates of the cross product; replacing x, y, z by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ gives us

$$\begin{aligned}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} &= (a_2b_3 - b_2a_3)\mathbf{i} - (a_1b_3 - b_1a_3)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k} \\ &= (a_2b_3 - b_2a_3)\mathbf{i} + (b_1a_3 - a_1b_3)\mathbf{j} + (a_1b_2 - b_1a_2)\mathbf{k} \\ &= \langle a_2b_3 - b_2a_3, b_1a_3 - a_1b_3, a_1b_2 - b_1a_2\rangle \\ &= \mathbf{A} \times \mathbf{B}.\end{aligned}$$

Given $\mathbf{A}$ and $\mathbf{B}$, there are typically two possible directions and an infinite number of magnitudes that will give a vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. As we have picked a particular one, we should investigate the magnitude and direction.

We know how to compute the magnitude of $\mathbf{A} \times \mathbf{B}$; it's a bit messy but not difficult. It is somewhat easier to work initially with the square of the magnitude, so as to avoid the square root:

$$\begin{aligned}|\mathbf{A} \times \mathbf{B}|^2 &= (a_2b_3 - b_2a_3)^2 + (b_1a_3 - a_1b_3)^2 + (a_1b_2 - b_1a_2)^2 \\ &= a_2^2b_3^2 - 2a_2b_3b_2a_3 + b_2^2a_3^2 + b_1^2a_3^2 - 2b_1a_3a_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1b_2b_1a_2 + b_1^2a_2^2\end{aligned}$$

While it is far from obvious, this nasty looking expression can be simplified:

$$\begin{aligned}|\mathbf{A} \times \mathbf{B}|^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\mathbf{A}|^2|\mathbf{B}|^2 - (\mathbf{A} \cdot \mathbf{B})^2 \\ &= |\mathbf{A}|^2|\mathbf{B}|^2 - |\mathbf{A}|^2|\mathbf{B}|^2\cos^2\theta \\ &= |\mathbf{A}|^2|\mathbf{B}|^2(1 - \cos^2\theta) \\ &= |\mathbf{A}|^2|\mathbf{B}|^2\sin^2\theta \\ |\mathbf{A} \times \mathbf{B}| &= |\mathbf{A}||\mathbf{B}|\sin\theta\end{aligned}$$

The magnitude of $\mathbf{A}\times\mathbf{B}$ is thus very similar to the dot product. In particular, notice that if $\mathbf{A}$ is parallel to $\mathbf{B}$, the angle between them is zero, so $\sin\theta = 0$, so $|\mathbf{A}\times\mathbf{B}| = 0$, and likewise if they are anti-parallel, $\sin\theta = 0$, and $|\mathbf{A}\times\mathbf{B}| = 0$. Conversely, if $|\mathbf{A}\times\mathbf{B}| = 0$ and $|\mathbf{A}|$ and $|\mathbf{B}|$ are not zero, it must be that $\sin\theta = 0$, so $\mathbf{A}$ is parallel or anti-parallel to $\mathbf{B}$.

Here is a curious fact about this quantity that turns out to be quite useful later on: Given two vectors, we can put them tail to tail and form a parallelogram, as in figure 12.4.1. The height of the parallelogram, h, is $|\mathbf{A}|\sin\theta$, and the base is $|\mathbf{B}|$, so the area of the parallelogram is $|\mathbf{A}||\mathbf{B}|\sin\theta$, exactly the magnitude of $|\mathbf{A}\times\mathbf{B}|$.

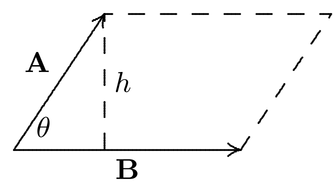


Figure 12.4.1 A parallelogram.

What about the direction of the cross product? Remarkably, there is a simple rule that describes the direction. Let's look at a simple example: Let $\mathbf{A} = \langle a,0,0\rangle$, $\mathbf{B} = \langle b,c,0\rangle$. If the vectors are placed with tails at the origin, $\mathbf{A}$ lies along the x-axis and $\mathbf{B}$ lies in the x-y plane, so we know the cross product will point either up or down. The cross product is

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & 0 & 0 \\ b & c & 0 \end{vmatrix} = \langle 0,0,ac\rangle.$$

As predicted, this is a vector pointing up or down, depending on the sign of ac. Suppose that $a > 0$, so the sign depends only on c: if $c > 0$, $ac > 0$ and the vector points up; if $c < 0$, the vector points down. On the other hand, if $a < 0$ and $c > 0$, the vector points down, while if $a < 0$ and $c < 0$, the vector points up. Here is how to interpret these facts with a single rule: Imagine rotating vector $\mathbf{A}$ until it points in the same direction as $\mathbf{B}$; there are two ways to do this—use the rotation that goes through the smaller angle. If $a > 0$ and $c > 0$, or $a < 0$ and $c < 0$, the rotation will be counter-clockwise when viewed from above; in the other two cases, $\mathbf{A}$ must be rotated clockwise to reach $\mathbf{B}$. The rule is: counter-clockwise means up, clockwise means down. If $\mathbf{A}$ and $\mathbf{B}$ are any vectors in the x-y plane, the same rule applies—$\mathbf{A}$ need not be parallel to the x-axis.

Although it is somewhat difficult computationally to see how this plays out for any two starting vectors, the rule is essentially the same. Place $\mathbf{A}$ and $\mathbf{B}$ tail to tail. The plane in which $\mathbf{A}$ and $\mathbf{B}$ lie may be viewed from two sides; view it from the side for which $\mathbf{A}$ must rotate counter-clockwise to reach $\mathbf{B}$; then the vector $\mathbf{A}\times\mathbf{B}$ points toward you.

This rule is usually called the **right hand rule**. Imagine placing the heel of your right hand at the point where the tails are joined, so that your slightly curled fingers indicate the direction of rotation from $\mathbf{A}$ to $\mathbf{B}$. Then your thumb points in the direction of the cross product $\mathbf{A} \times \mathbf{B}$.

One immediate consequence of these facts is that $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$, because the two cross products point in the opposite direction. On the other hand, since

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta = |\mathbf{B}||\mathbf{A}|\sin\theta = |\mathbf{B} \times \mathbf{A}|,$$

the lengths of the two cross products are equal, so we know that $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$.

The cross product has some familiar-looking properties that will be useful later, so we list them here. As with the dot product, these can be proved by performing the appropriate calculations on coordinates, after which we may sometimes avoid such calculations by using the properties.

THEOREM 12.4.1 If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and a is a real number, then

1. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
2. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
3. $(a\mathbf{u}) \times \mathbf{v} = a(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (a\mathbf{v})$
4. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
5. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ ■

Exercises 12.4.

1. Find the cross product of $\langle 1,1,1\rangle$ and $\langle 1,2,3\rangle$. ⇒
2. Find the cross product of $\langle 1,0,2\rangle$ and $\langle -1,-2,4\rangle$. ⇒
3. Find the cross product of $\langle -2,1,3\rangle$ and $\langle 5,2,-1\rangle$. ⇒
4. Find the cross product of $\langle 1,0,0\rangle$ and $\langle 0,0,1\rangle$. ⇒
5. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are separated by an angle of $\pi/6$, and $|\mathbf{u}| = 2$ and $|\mathbf{v}| = 3$. Find $|\mathbf{u} \times \mathbf{v}|$. ⇒
6. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are separated by an angle of $\pi/4$, and $|\mathbf{u}| = 3$ and $|\mathbf{v}| = 7$. Find $|\mathbf{u} \times \mathbf{v}|$. ⇒
7. Find the area of the parallelogram with vertices $(0,0)$, $(1,2)$, $(3,7)$, and $(2,5)$. ⇒
8. Find and explain the value of $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$ and $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$.
9. Prove that for all vectors $\mathbf{u}$ and $\mathbf{v}$, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$.
10. Prove Theorem 12.4.1.
11. Define the triple product of three vectors, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, to be the scalar $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$. Show that three vectors lie in the same plane if and only if their triple product is zero. Verify that $\langle 1,5,-2\rangle$, $\langle 4,3,0\rangle$ and $\langle 6,13,-4\rangle$ are coplanar.

12.5 Lines and Planes

Lines and planes are perhaps the simplest of curves and surfaces in three dimensional space. They also will prove important as we seek to understand more complicated curves and surfaces.

The equation of a line in two dimensions is $ax + by = c$; it is reasonable to expect that a line in three dimensions is given by $ax + by + cz = d$; reasonable, but wrong—it turns out that this is the equation of a plane.

A plane does not have an obvious "direction" as does a line. It is possible to associate a plane with a direction in a very useful way, however: there are exactly two directions perpendicular to a plane. Any vector with one of these two directions is called **normal** to the plane. So while there are many normal vectors to a given plane, they are all parallel or anti-parallel to each other.

Suppose two points (v_1, v_2, v_3) and (w_1, w_2, w_3) are in a plane; then the vector $\langle w_1 - v_1, w_2 - v_2, w_3 - v_3\rangle$ is parallel to the plane; in particular, if this vector is placed with its tail at (v_1, v_2, v_3) then its head is at (w_1, w_2, w_3) and it lies in the plane. As a result, any vector perpendicular to the plane is perpendicular to $\langle w_1 - v_1, w_2 - v_2, w_3 - v_3\rangle$. In fact, it is easy to see that the plane consists of *precisely* those points (w_1, w_2, w_3) for which $\langle w_1 - v_1, w_2 - v_2, w_3 - v_3\rangle$ is perpendicular to a normal to the plane, as indicated in figure 12.5.1. Turning this around, suppose we know that $\langle a, b, c\rangle$ is normal to a plane containing the point (v_1, v_2, v_3). Then (x, y, z) is in the plane if and only if $\langle a, b, c\rangle$ is perpendicular to $\langle x - v_1, y - v_2, z - v_3\rangle$. In turn, we know that this is true precisely when $\langle a, b, c\rangle \cdot \langle x - v_1, y - v_2, z - v_3\rangle = 0$. That is, (x, y, z) is in the plane if and only if

$$\begin{aligned}
\langle a, b, c\rangle \cdot \langle x - v_1, y - v_2, z - v_3\rangle &= 0\\
a(x - v_1) + b(y - v_2) + c(z - v_3) &= 0\\
ax + by + cz - av_1 - bv_2 - cv_3 &= 0\\
ax + by + cz &= av_1 + bv_2 + cv_3.
\end{aligned}$$

Working backwards, note that if (x, y, z) is a point satisfying $ax + by + cz = d$ then

$$\begin{aligned}
ax + by + cz &= d\\
ax + by + cz - d &= 0\\
a(x - d/a) + b(y - 0) + c(z - 0) &= 0\\
\langle a, b, c\rangle \cdot \langle x - d/a, y, z\rangle &= 0.
\end{aligned}$$

Namely, $\langle a, b, c\rangle$ is perpendicular to the vector with tail at $(d/a, 0, 0)$ and head at (x, y, z). This means that the points (x, y, z) that satisfy the equation $ax + by + cz = d$ form a

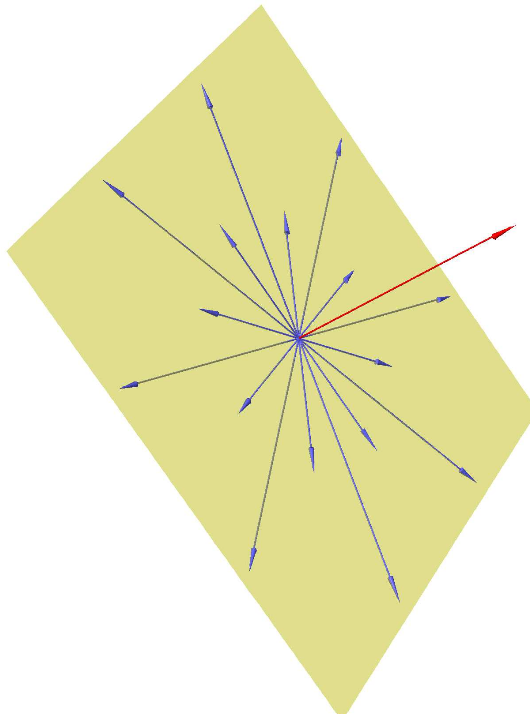


Figure 12.5.1 A plane defined via vectors perpendicular to a normal. (AP)

plane perpendicular to $\langle a, b, c\rangle$. (This doesn't work if $a = 0$, but in that case we can use b or c in the role of a. That is, either $a(x - 0) + b(y - d/b) + c(z - 0) = 0$ or $a(x - 0) + b(y - 0) + c(z - d/c) = 0$.)

Thus, given a vector $\langle a, b, c\rangle$ we know that all planes perpendicular to this vector have the form $ax + by + cz = d$, and any surface of this form is a plane perpendicular to $\langle a, b, c\rangle$.

EXAMPLE 12.5.1 Find an equation for the plane perpendicular to $\langle 1, 2, 3\rangle$ and containing the point $(5, 0, 7)$.

Using the derivation above, the plane is $1x + 2y + 3z = 1 \cdot 5 + 2 \cdot 0 + 3 \cdot 7 = 26$. Alternately, we know that the plane is $x + 2y + 3z = d$, and to find d we may substitute the known point on the plane to get $5 + 2 \cdot 0 + 3 \cdot 7 = d$, so $d = 26$. □

EXAMPLE 12.5.2 Find a vector normal to the plane $2x - 3y + z = 15$.

One example is $\langle 2, -3, 1\rangle$. Any vector parallel or anti-parallel to this works as well, so for example $-2\langle 2, -3, 1\rangle = \langle -4, 6, -2\rangle$ is also normal to the plane. □

We will frequently need to find an equation for a plane given certain information about the plane. While there may occasionally be slightly shorter ways to get to the desired result, it is always possible, and usually advisable, to use the given information to find a normal to the plane and a point on the plane, and then to find the equation as above.

EXAMPLE 12.5.3 The planes $x - z = 1$ and $y + 2z = 3$ intersect in a line. Find a third plane that contains this line and is perpendicular to the plane $x + y - 2z = 1$.

First, we note that two planes are perpendicular if and only if their normal vectors are perpendicular. Thus, we seek a vector $\langle a, b, c \rangle$ that is perpendicular to $\langle 1, 1, -2 \rangle$. In addition, since the desired plane is to contain a certain line, $\langle a, b, c \rangle$ must be perpendicular to any vector parallel to this line. Since $\langle a, b, c \rangle$ must be perpendicular to two vectors, we may find it by computing the cross product of the two. So we need a vector parallel to the line of intersection of the given planes. For this, it suffices to know two points on the line. To find two points on this line, we must find two points that are simultaneously on the two planes, $x - z = 1$ and $y + 2z = 3$. Any point on both planes will satisfy $x - z = 1$ and $y + 2z = 3$. It is easy to find values for x and z satisfying the first, such as $x = 1, z = 0$ and $x = 2, z = 1$. Then we can find corresponding values for y using the second equation, namely $y = 3$ and $y = 1$, so $(1, 3, 0)$ and $(2, 1, 1)$ are both on the line of intersection because both are on both planes. Now $\langle 2-1, 1-3, 1-0 \rangle = \langle 1, -2, 1 \rangle$ is parallel to the line. Finally, we may choose $\langle a, b, c \rangle = \langle 1, 1, -2 \rangle \times \langle 1, -2, 1 \rangle = \langle -3, -3, -3 \rangle$. While this vector will do perfectly well, any vector parallel or anti-parallel to it will work as well, so for example we might choose $\langle 1, 1, 1 \rangle$ which is anti-parallel to it.

Now we know that $\langle 1, 1, 1 \rangle$ is normal to the desired plane and $(2, 1, 1)$ is a point on the plane. Therefore an equation of the plane is $x + y + z = 4$. As a quick check, since $(1, 3, 0)$ is also on the line, it should be on the plane; since $1 + 3 + 0 = 4$, we see that this is indeed the case.

Note that had we used $\langle -3, -3, -3 \rangle$ as the normal, we would have discovered the equation $-3x - 3y - 3z = -12$, then we might well have noticed that we could divide both sides by -3 to get the equivalent $x + y + z = 4$. □

So we now understand equations of planes; let us turn to lines. Unfortunately, it turns out to be quite inconvenient to represent a typical line with a single equation; we need to approach lines in a different way.

Unlike a plane, a line in three dimensions does have an obvious direction, namely, the direction of any vector parallel to it. In fact a line can be defined and uniquely identified by providing one point on the line and a vector parallel to the line (in one of two possible directions). That is, the line consists of exactly those points we can reach by starting at the point and going for some distance in the direction of the vector. Let's see how we can translate this into more mathematical language.

Suppose a line contains the point (v_1, v_2, v_3) and is parallel to the vector $\langle a, b, c \rangle$. If we place the vector $\langle v_1, v_2, v_3 \rangle$ with its tail at the origin and its head at (v_1, v_2, v_3), and if we place the vector $\langle a, b, c \rangle$ with its tail at (v_1, v_2, v_3), then the head of $\langle a, b, c \rangle$ is at a point on the line. We can get to *any* point on the line by doing the same thing, except using $t\langle a, b, c \rangle$ in place of $\langle a, b, c \rangle$, where t is some real number. Because of the way vector

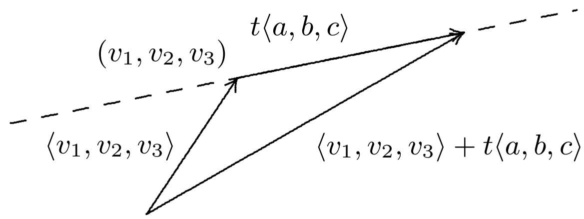


Figure 12.5.2 Vector form of a line.

addition works, the point at the head of the vector $t\langle a,b,c\rangle$ is the point at the head of the vector $\langle v_1,v_2,v_3\rangle + t\langle a,b,c\rangle$, namely (v_1+ta, v_2+tb, v_3+tc); see figure 12.5.2.

In other words, as t runs through all possible real values, the vector $\langle v_1,v_2,v_3\rangle + t\langle a,b,c\rangle$ points to every point on the line when its tail is placed at the origin. Another common way to write this is as a set of **parametric equations**:

$$x = v_1 + ta \qquad y = v_2 + tb \qquad z = v_3 + tc.$$

It is occasionally useful to use this form of a line even in two dimensions; a vector form for a line in the x-y plane is $\langle v_1,v_2\rangle + t\langle a,b\rangle$, which is the same as $\langle v_1,v_2,0\rangle + t\langle a,b,0\rangle$.

EXAMPLE 12.5.4 Find a vector expression for the line through $(6,1,-3)$ and $(2,4,5)$. To get a vector parallel to the line we subtract $\langle 6,1,-3\rangle - \langle 2,4,5\rangle = \langle 4,-3,-8\rangle$. The line is then given by $\langle 2,4,5\rangle + t\langle 4,-3,-8\rangle$; there are of course many other possibilities, such as $\langle 6,1,-3\rangle + t\langle 4,-3,-8\rangle$. □

EXAMPLE 12.5.5 Determine whether the lines $\langle 1,1,1\rangle + t\langle 1,2,-1\rangle$ and $\langle 3,2,1\rangle + t\langle -1,-5,3\rangle$ are parallel, intersect, or neither.

In two dimensions, two lines either intersect or are parallel; in three dimensions, lines that do not intersect might not be parallel. In this case, since the direction vectors for the lines are not parallel or anti-parallel we know the lines are not parallel. If they intersect, there must be two values a and b so that $\langle 1,1,1\rangle + a\langle 1,2,-1\rangle = \langle 3,2,1\rangle + b\langle -1,-5,3\rangle$, that is,

$$1 + a = 3 - b$$
$$1 + 2a = 2 - 5b$$
$$1 - a = 1 + 3b$$

This gives three equations in two unknowns, so there may or may not be a solution in general. In this case, it is easy to discover that $a = 3$ and $b = -1$ satisfies all three equations, so the lines do intersect at the point $(4,7,-2)$. □

EXAMPLE 12.5.6 Find the distance from the point $(1,2,3)$ to the plane $2x-y+3z = 5$. The distance from a point P to a plane is the shortest distance from P to any point on the plane; this is the distance measured from P perpendicular to the plane; see figure 12.5.3. This distance is the absolute value of the scalar projection of $\overrightarrow{QP}$ onto a normal vector $\mathbf{n}$, where Q is any point on the plane. It is easy to find a point on the plane, say $(1,0,1)$. Thus the distance is

$$\frac{\overrightarrow{QP}\cdot\mathbf{n}}{|\mathbf{n}|}=\frac{\langle 0,2,2\rangle\cdot\langle 2,-1,3\rangle}{|\langle 2,-1,3\rangle|}=\frac{4}{\sqrt{14}}.$$

□

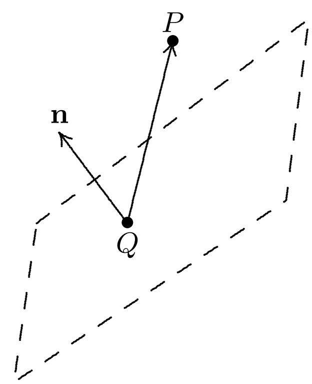


Figure 12.5.3 Distance from a point to a plane.

EXAMPLE 12.5.7 Find the distance from the point $(-1,2,1)$ to the line $\langle 1,1,1\rangle + t\langle 2,3,-1\rangle$. Again we want the distance measured perpendicular to the line, as indicated in figure 12.5.4. The desired distance is

$$|\overrightarrow{QP}|\sin\theta=\frac{|\overrightarrow{QP}\times\mathbf{A}|}{|\mathbf{A}|},$$

where $\mathbf{A}$ is any vector parallel to the line. From the equation of the line, we can use $Q=(1,1,1)$ and $\mathbf{A}=\langle 2,3,-1\rangle$, so the distance is

$$\frac{|\langle -2,1,0\rangle\times\langle 2,3,-1\rangle|}{\sqrt{14}}=\frac{|\langle -1,-2,-8\rangle|}{\sqrt{14}}=\frac{\sqrt{69}}{\sqrt{14}}.$$

□

Exercises 12.5.

1. Find an equation of the plane containing $(6,2,1)$ and perpendicular to $\langle 1,1,1\rangle$. ⇒
2. Find an equation of the plane containing $(-1,2,-3)$ and perpendicular to $\langle 4,5,-1\rangle$. ⇒
3. Find an equation of the plane containing $(1,2,-3)$, $(0,1,-2)$ and $(1,2,-2)$. ⇒

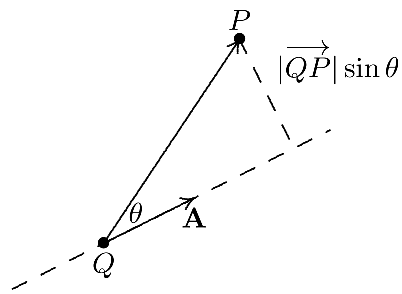


Figure 12.5.4 Distance from a point to a line.

4. Find an equation of the plane containing $(1,0,0)$, $(4,2,0)$ and $(3,2,1)$. ⇒

5. Find an equation of the plane containing $(1,0,0)$ and the line $\langle 1,0,2\rangle + t\langle 3,2,1\rangle$. ⇒

6. Find an equation of the plane containing the line of intersection of $x+y+z=1$ and $x-y+2z=2$, and perpendicular to the x-y plane. ⇒

7. Find an equation of the line through $(1,0,3)$ and $(1,2,4)$. ⇒

8. Find an equation of the line through $(1,0,3)$ and perpendicular to the plane $x+2y-z=1$. ⇒

9. Find an equation of the line through the origin and perpendicular to the plane $x+y-z=2$. ⇒

10. Find a and c so that $(a,1,c)$ is on the line through $(0,2,3)$ and $(2,7,5)$. ⇒

11. Explain how to discover the solution in example 12.5.5.

12. Determine whether the lines $\langle 1,3,-1\rangle + t\langle 1,1,0\rangle$ and $\langle 0,0,0\rangle + t\langle 1,4,5\rangle$ are parallel, intersect, or neither. ⇒

13. Determine whether the lines $\langle 1,0,2\rangle + t\langle -1,-1,2\rangle$ and $\langle 4,4,2\rangle + t\langle 2,2,-4\rangle$ are parallel, intersect, or neither. ⇒

14. Determine whether the lines $\langle 1,2,-1\rangle + t\langle 1,2,3\rangle$ and $\langle 1,0,1\rangle + t\langle 2/3,2,4/3\rangle$ are parallel, intersect, or neither. ⇒

15. Determine whether the lines $\langle 1,1,2\rangle + t\langle 1,2,-3\rangle$ and $\langle 2,3,-1\rangle + t\langle 2,4,-6\rangle$ are parallel, intersect, or neither. ⇒

16. Find a unit normal vector to each of the coordinate planes.

17. Show that $\langle 2,1,3\rangle + t\langle 1,1,2\rangle$ and $\langle 3,2,5\rangle + s\langle 2,2,4\rangle$ are the same line.

18. Give a prose description for each of the following processes:

- **a.** Given two distinct points, find the line that goes through them.
- **b.** Given three points (not all on the same line), find the plane that goes through them. Why do we need the caveat that not all points be on the same line?
- **c.** Given a line and a point not on the line, find the plane that contains them both.
- **d.** Given a plane and a point not on the plane, find the line that is perpendicular to the plane through the given point.

19. Find the distance from $(2,2,2)$ to $x+y+z=-1$. ⇒

20. Find the distance from $(2,-1,-1)$ to $2x-3y+z=2$. ⇒

21. Find the distance from $(2,-1,1)$ to $\langle 2,2,0\rangle + t\langle 1,2,3\rangle$. ⇒

22. Find the distance from $(1,0,1)$ to $\langle 3,2,1\rangle + t\langle 2,-1,-2\rangle$. ⇒

23. Find the cosine of the angle between the planes $x+y+z=2$ and $x+2y+3z=8$. ⇒

24. Find the cosine of the angle between the planes $x-y+2z=2$ and $3x-2y+z=5$. ⇒

12.6 Other Coordinate Systems

Coordinate systems are tools that let us use algebraic methods to understand geometry. While the **rectangular** (also called **Cartesian**) coordinates that we have been discussing are the most common, some problems are easier to analyze in alternate coordinate systems.

A coordinate system is a scheme that allows us to identify any point in the plane or in three-dimensional space by a set of numbers. In rectangular coordinates these numbers are interpreted, roughly speaking, as the lengths of the sides of a rectangular "box."

In two dimensions you may already be familiar with an alternative, called **polar coordinates**. In this system, each point in the plane is identified by a pair of numbers (r,θ). The number θ measures the angle between the positive x-axis and a vector with tail at the origin and head at the point, as shown in figure 12.6.1; the number r measures the distance from the origin to the point. Either of these may be negative; a negative θ indicates the angle is measured clockwise from the positive x-axis instead of counter-clockwise, and a negative r indicates the point at distance $|r|$ in the opposite of the direction given by θ. Figure 12.6.1 also shows the point with rectangular coordinates $(1,\sqrt{3})$ and polar coordinates $(2,\pi/3)$, 2 units from the origin and $\pi/3$ radians from the positive x-axis.

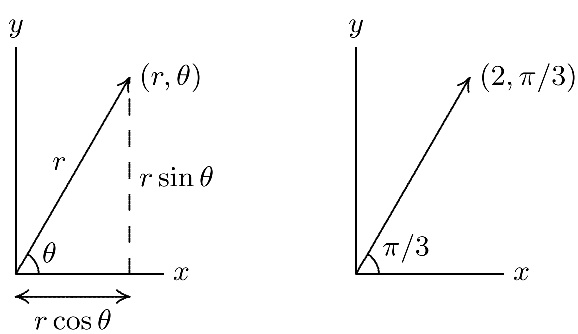


Figure 12.6.1 Polar coordinates: the general case and the point with rectangular coordinates $(1,\sqrt{3})$.

We can extend polar coordinates to three dimensions simply by adding a z coordinate; this is called **cylindrical coordinates**. Each point in three-dimensional space is represented by three coordinates (r,θ,z) in the obvious way: this point is z units above or below the point (r,θ) in the x-y plane, as shown in figure 12.6.2. The point with rectangular coordinates $(1,\sqrt{3},3)$ and cylindrical coordinates $(2,\pi/3,3)$ is also indicated in figure 12.6.2.

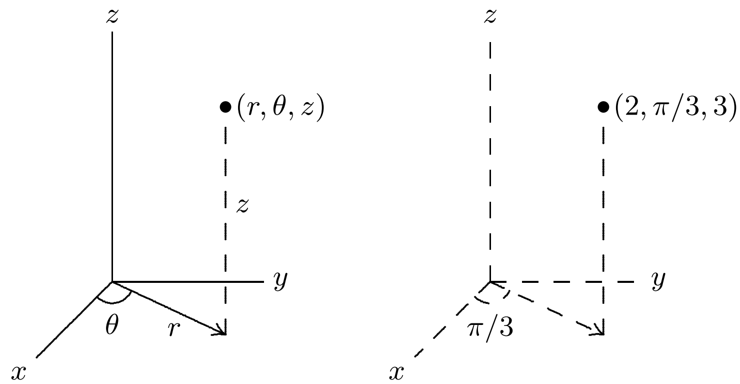


Figure 12.6.2 Cylindrical coordinates: the general case and the point with rectangular coordinates $(1, \sqrt{3}, 3)$.

Some figures with relatively complicated equations in rectangular coordinates will be represented by simpler equations in cylindrical coordinates. For example, the cylinder in figure 12.6.3 has equation $x^2 + y^2 = 4$ in rectangular coordinates, but equation $r = 2$ in cylindrical coordinates.

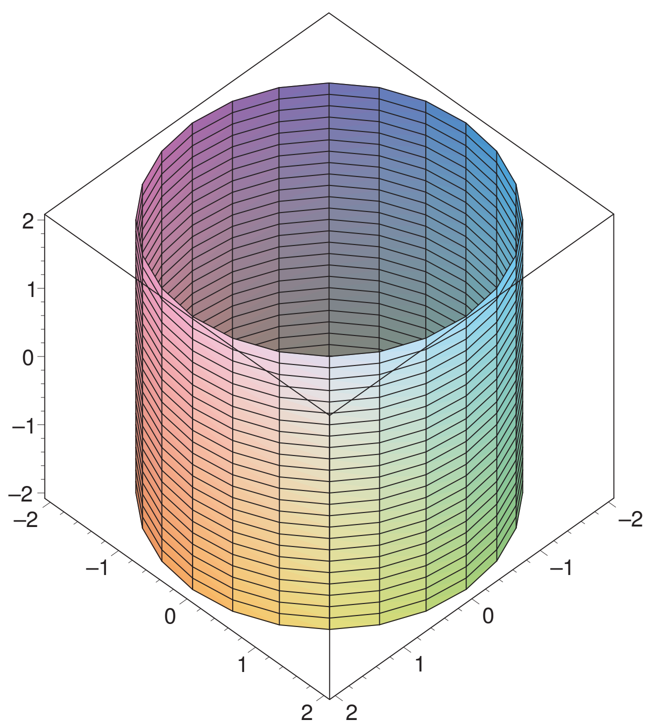


Figure 12.6.3 The cylinder $r = 2$.

Given a point (r, θ) in polar coordinates, it is easy to see (as in figure 12.6.1) that the rectangular coordinates of the same point are $(r\cos\theta, r\sin\theta)$, and so the point (r, θ, z) in cylindrical coordinates is $(r\cos\theta, r\sin\theta, z)$ in rectangular coordinates. This means it is usually easy to convert any equation from rectangular to cylindrical coordinates: simply substitute

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and leave z alone. For example, starting with $x^2 + y^2 = 4$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ gives

$$\begin{aligned} r^2\cos^2\theta + r^2\sin^2\theta &= 4 \\ r^2(\cos^2\theta + \sin^2\theta) &= 4 \\ r^2 &= 4 \\ r &= 2. \end{aligned}$$

Of course, it's easy to see directly that this defines a cylinder as mentioned above.

Cylindrical coordinates are an obvious extension of polar coordinates to three dimensions, but the use of the z coordinate means they are not as closely analogous to polar coordinates as another standard coordinate system. In polar coordinates, we identify a point by a direction and distance from the origin; in three dimensions we can do the same thing, in a variety of ways. The question is: how do we represent a direction? One way is to give the angle of rotation, θ, from the positive x axis, just as in cylindrical coordinates, and also an angle of rotation, ϕ, from the positive z axis. Roughly speaking, θ is like longitude and ϕ is like latitude. (Earth longitude is measured as a positive or negative angle from the prime meridian, and is always between 0 and 180 degrees, east or west; θ can be any positive or negative angle, and we use radians except in informal circumstances. Earth latitude is measured north or south from the equator; ϕ is measured from the north pole down.) This system is called **spherical coordinates**; the coordinates are listed in the order (ρ, θ, ϕ), where ρ is the distance from the origin, and like r in cylindrical coordinates it may be negative. The general case and an example are pictured in figure 12.6.4; the length marked r is the r of cylindrical coordinates.

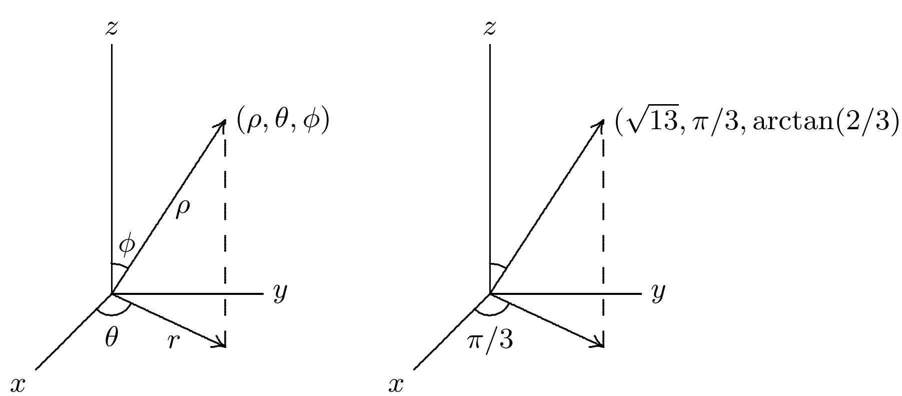


Figure 12.6.4 Spherical coordinates: the general case and the point with rectangular coordinates $(1, \sqrt{3}, 3)$.

As with cylindrical coordinates, we can easily convert equations in rectangular coordinates to the equivalent in spherical coordinates, though it is a bit more difficult to discover the proper substitutions. Figure 12.6.5 shows the typical point in spherical coordinates from figure 12.6.4, viewed now so that the arrow marked r in the original graph appears as the horizontal "axis" in the left hand graph. From this diagram it is easy to see that the z coordinate is $\rho\cos\phi$, and that $r = \rho\sin\phi$, as shown. Thus, in converting from rectangular to spherical coordinates we will replace z by $\rho\cos\phi$. To see the substitutions for x and y we now view the same point from above, as shown in the right hand graph. The hypotenuse of the triangle in the right hand graph is $r = \rho\sin\phi$, so the sides of the triangle, as shown, are $x = r\cos\theta = \rho\sin\phi\cos\theta$ and $y = r\sin\theta = \rho\sin\phi\sin\theta$. So the upshot is that to convert from rectangular to spherical coordinates, we make these substitutions:

$$
\begin{aligned}
x &= \rho\sin\phi\cos\theta \\
y &= \rho\sin\phi\sin\theta \\
z &= \rho\cos\phi.
\end{aligned}
$$

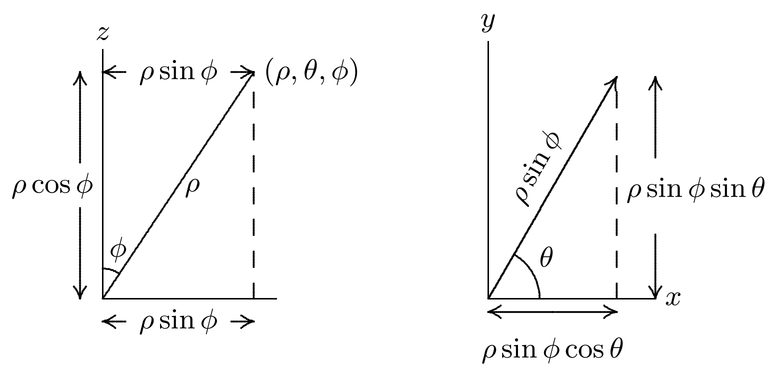


Figure 12.6.5 Converting from rectangular to spherical coordinates.

EXAMPLE 12.6.1 As the cylinder had a simple equation in cylindrical coordinates, so does the sphere in spherical coordinates: $\rho = 2$ is the sphere of radius 2. If we start with the Cartesian equation of the sphere and substitute, we get the spherical equation:

$$\begin{aligned} x^2 + y^2 + z^2 &= 2^2 \\ \rho^2 \sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi &= 2^2 \\ \rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho^2\cos^2\phi &= 2^2 \\ \rho^2\sin^2\phi + \rho^2\cos^2\phi &= 2^2 \\ \rho^2(\sin^2\phi + \cos^2\phi) &= 2^2 \\ \rho^2 &= 2^2 \\ \rho &= 2 \end{aligned}$$

□

EXAMPLE 12.6.2 Find an equation for the cylinder $x^2 + y^2 = 4$ in spherical coordinates.

Proceeding as in the previous example:

$$\begin{aligned} x^2 + y^2 &= 4 \\ \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta &= 4 \\ \rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) &= 4 \\ \rho^2\sin^2\phi &= 4 \\ \rho\sin\phi &= 2 \\ \rho &= \frac{2}{\sin\phi} \end{aligned}$$

□

Exercises 12.6.

1. Convert the following points in rectangular coordinates to cylindrical and spherical coordinates:
   - **a.** $(1, 1, 1)$
   - **b.** $(7, -7, 5)$
   - **c.** $(\cos(1), \sin(1), 1)$
   - **d.** $(0, 0, -\pi)$ ⇒
2. Find an equation for the sphere $x^2 + y^2 + z^2 = 4$ in cylindrical coordinates. ⇒
3. Find an equation for the y-z plane in cylindrical coordinates. ⇒
4. Find an equation equivalent to $x^2 + y^2 + 2z^2 + 2z - 5 = 0$ in cylindrical coordinates. ⇒
5. Suppose the curve $z = e^{-x^2}$ in the x-z plane is rotated around the z axis. Find an equation for the resulting surface in cylindrical coordinates. ⇒

6. Suppose the curve $z = x$ in the x-z plane is rotated around the z axis. Find an equation for the resulting surface in cylindrical coordinates. ⇒

7. Find an equation for the plane $y = 0$ in spherical coordinates. ⇒

8. Find an equation for the plane $z = 1$ in spherical coordinates. ⇒

9. Find an equation for the sphere with radius 1 and center at $(0, 1, 0)$ in spherical coordinates. ⇒

10. Find an equation for the cylinder $x^2 + y^2 = 9$ in spherical coordinates. ⇒

11. Suppose the curve $z = x$ in the x-z plane is rotated around the z axis. Find an equation for the resulting surface in spherical coordinates. ⇒

12. Plot the polar equations $r = \sin(\theta)$ and $r = \cos(\theta)$ and comment on their similarities. (If you get stuck on how to plot these, you can multiply both sides of each equation by r and convert back to rectangular coordinates).

13. Extend exercises 6 and 11 by rotating the curve $z = mx$ around the z axis and converting to both cylindrical and spherical coordinates. ⇒

14. Convert the spherical formula $\rho = \sin\theta\sin\phi$ to rectangular coordinates and describe the surface defined by the formula (Hint: Multiply both sides by ρ.) ⇒

15. We can describe points in the first octant by $x > 0$, $y > 0$ and $z > 0$. Give similar inequalities for the first octant in cylindrical and spherical coordinates. ⇒